IDENTIFICATION AND OPTIMIZATION AS MAIN APPLICATION AREAS FOR MATHEMATICAL METHODS IN FUSE DESIGN

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The efficiency of experimental approaches and modelling techniques in the design of modern quick-acting fuses could be greatly increased if the variety of problems arising in the field were tractable in terms of identification and optimization. Identification is understood as a search for qualitative and quantitative relations between structural parameters and performance characteristics of the fuse, either through the use of a known mathematical model or in the course of developing one. The relations found form a basis for predictions. The stage of critical importance for identification is that of elaborating the model structure and evaluating model parameters. The necessity for such a stage in the fuse design is conditioned by the fact that fuses, unlike electromagnetic devices for instance, do not show strict analytical relationships between structural parameters and performance characteristics. In the case of fuses, the relations whose determination can be regarded as classical identification problems include those between

a) the temperature field in the fuse and structural parameters;

b) the time-current relation and structural parameters;

c) cycling capacity and structural parameters and/or type of repeated loading and

d) functional characteristics (like arcing current, $\int i_a^2 dt$ and $\int i_a U_a dt$ functionals, etc.) and structural parameters or those of the short circuit.

Depending on which parameters are known, these problems can be formulated either as direct or inverse. As a direct problem, problem a) can be analyzed, with allowance for nonlinearity and complex geometry of the boundary surfaces, in the variational formulation, using the finite-element technique[1].

In case the choice of the functional for the finite-element model is difficult, or the proper functional cannot be obtained at all, then a useful finite-element formulation is provided by Galerkin's scheme. Galerkin suggested a method for solving boundary-value problems through an approximate solution of a partial differential equation[2]. If applied to the field equation characteristic of fuse design problems, i.e.

$$L(\varphi)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+Q=0, \qquad (1)$$

then Galerkin's finite-element formulation leads to

$$\iint_D [B]^T[B]\,dy\,dx\{\Phi\}-\iint_D [N]^T Q\,dx\,dy, \qquad (2)$$

where $\varphi=[N]\{\Phi\}$ is the finite-element approximation for the unknown function, and

$$[B]^T[B]=\frac{\partial [N]^T}{\partial x}\cdot\frac{\partial [N]}{\partial x}+\frac{\partial [N]^T}{\partial y}\cdot\frac{\partial [N]}{\partial y}. \qquad (3)$$

The further procedure is similar to that applied to variationally formulated finite-element equations. If combined with a polynomial approximation of the functions involved, Galerkin's technique provides a highly efficient, simple and general method for solving the equations and boundary-value problems encountered in the fuse design.

Identification problem b) can be diversified, depending on various conditions. In the range of extremely high short-circuit currents, (300 to 500)I_n, the analysis can be based on Meyer's equation disregarding the geometry of constrictions, waveform of the current and the thermal flux from the necks to the wider parts of the fuse element and the filler. If the current values are so high that constricted parts get fused over 14 or 15 ms, then the calculation may be carried out without allowance for the heat transfer to the filler. Indeed, our experiments with both filled and unfilled fuses showed the same value of the fusion integral for all specimens. In case the fusion time exceeds 15 ms, a complete finite-element calculation is required. In certain cases, it proves necessary to consider the rather interesting problem of select such values of the structure para-

meter with which its time-current characteristic would pass through at least four given points. The solution procedure is close to that of optimization.

No rigorous models are known for problems c) and d), generally the model structure needs to be developed from basic physical principles (especially for new, non-standard conditions). In many particular cases, however, some a priori knowledge can be used efficiently to simplify the model structure. Then evaluation of the model parameters, which stage is necessary because of the many random factors influencing the results of numerical or instrumental experiments, is reduced to approximation. The latter problem can be efficiently analyzed for the case of fuses within the approach suggested by the present authors and employing the Kolmogorov-Mordashov transformation. As was known by A.N. Kolmogorov, an unknown function $f(x_1, x_2, \ldots, x_n)$ of many variables can be represented, for the purpose of approximation, as a superposition of single-variable functions. V.M. Mordashov further suggested using some transformation rather than a simple superposition of single-variable functions, i.e.

$$f(x_1, x_2, \ldots, x_n) = \mathcal{L}^{-1}\left\{ \sum_{i=1}^{n} \varphi_i(x_i) \right\}, \qquad (4)$$

where the values assumed by the wanted function $f(\bar{x})$ at certain points are known and $\mathcal{L}(f) = \sum_{i=1}^{s} \alpha_i f_i = \sum_{i=1}^{n} \varphi_i(x_i)$ is a transformation of $f(x)$. The unknown parameters of the transformation are α_i and φ_i, where the vector $\vec{\alpha}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is to be found as the eigenvector minimizing the matrix form $[\Phi - \Psi - \lambda N] = 0$, with

$$\Phi = \left\| \int_{\mathcal{D}} \varphi_i(G_n)\varphi_j(G_n)\, d\chi(G_n) \right\|_1^K$$
$$\Psi = \left\| \int_{\mathcal{D}} \psi_i(G_n)\psi_j(G_n)\, d\chi(G_n) \right\|_1^K \qquad (5)$$

$G_n = \{x_1, x_2, \ldots x_n\}$ is the set of independent variables in $\mathcal{D}$ $(G_n \in \mathcal{D})$;
$d\chi(G_n) = \prod_{i=1}^{n} d\chi(x_i)$; $d\chi_i(x_i) = 1/p$, where p is the number of variation levels for the argument; λ is the matrix parameter; N a positively defined symmetric matrix of order K to allow for the importance (weight) of the function arguments, and φ_i a set of arbitrarily selected linear-independent functions.

The fundamental advantage of the approach described is the possibility of obtaining a realistic model and improving the accuracy, without increasing the volume of the instrumental experiment, solely at the expense of sophisticated mathematical transformations. The current practical models for solving identification problem c) take the simplest structure at the empirical level (more details can be found in[3]). Among the approaches to identification problem d), one appeals to abstract, idealized models employing a specific preset time-voltage characteristic of the arc. The present authors have performed a series of computer calculations of basic performance characteristics of fuses, using the simplest voltage waveforms across the fuse arc. Generally, the arc voltage can be represented as

$$U_g(t) = 2(U_{go} - U_{gi})T/T_K + U_{gi}, \qquad (6)$$

where U_{gi} is equal to U_{gmin} for the waveform $U_{\Delta i}$, U_{go} for the mean value and U_{gmax} for U_{pi}. The four waveforms used here are $U_{\Delta i}$ which is voltage reaching the level U_{gmin} instantaneously and then increasing linearly to U_{gmax}; U_{pi} which becomes U_{gmax} instantaneously and then decreases in a linear manner towards the level U_{gmin} (or U_c) of the power line voltage; U_{go} is the voltage reaching its value instantaneously and staying at that level during the entire arc extinguishing stage; the same value is the mean of $U_{\Delta i}$ and U_{pi}; and finally U_{gmax} is a constant voltage equal to the maximum admissible level. For the sake of simplicity, we considered the case of an interrupted d.c., assuming that arcing started at the moment t_o when the current reached the fusion value I_o. The calculations were performed for U_{go} =600 to 1000 V; I_o =10 to 20 kA; U_c =500 V; L =(5×10^{-5} to 6×10^{-4}) H, and R=5×10^{-3} to 2×10^{-2} Ohm. Now we describe some of the regularities established.

The arcing time is determined by the mean voltage across the arc, being independent of the voltage waveform. At $U_g = U_{gmax}$ = =const it is 1.5 times shorter than with other voltage types. Whereas the arc Joule integral and the arc energy both depend

on the voltage waveform across the arc, changing by a factor of 2 or 3 for different waveforms, even with the same mean voltage. The dependence upon this latter is quite essential. E.g. a change of U_{go} from $2U_c$ to $1{,}2U_c$ results in a nearly five-fold increase in the arc integral. Besides, 90 % of the arc energy at $U_{go}=5U_c$ consist of the energy accumulated in the inductor, while with $U_{go}=1{,}2U_c$ this part of energy makes up only 20 to 25 % of the total energy.

Optimization is understood as a search for the vector of adjustable fuse parameters minimizing some goal (quality) function with specific functional and/or parametric constraints. All the above mentioned identification problems can be formulated as optimization problems as well. Of the many optimization approaches known for fuses we describe some that have been recognized as the most efficient. One-dimensional optimization with Fibonacci numbers. This is a sequential method of searching, e.g. for a current providing a maximum arc energy, in which the preceding experimental results influence the choice of the domain for the next experiment. In mathematical terms, the problem can be formulated as minimization of the error ε with which the point x^* is determined, minimizing the wanted function $f(x)$ over the interval $[a_k, b_k]$ after n steps of the process. According to the method, we select points x_k and x_k' within $[a_k, b_k]$, determined by the equations

$$x_k = \frac{F_{N-1-k}}{F_{N+1-k}}(b_k - a_k) + a_k \qquad (7)$$

and

$$x_k' = \frac{F_{N-1}}{F_{N+1-k}}(b_k - a_k) + a_k, \qquad (8)$$

where $F_k = F_{k-1} + F_{k-2}$ are Fibonacci numbers, with $F_0 = F_1 = 1$. Thus, at the first step we consider the interval $[a_k, b_k]$ which is the entire domain of the function. Further steps result in a gradual contraction of the interval, following shifts of one of the boundaries (either the right- or the left-hand one) at the preceding step. If the function assumes at x_k a smaller value than at x_k' $f(x_k) < f(x_k')$, then the next uncertainty interval can be selected as $[a_{k+1}\, b_{k+1}] = [a_k\, x_k']$.

Contrary to this, with $f(x_k) > f(x_k')$ the interval to be analysed is $[a_{k+1}\, b_{k+1}] = [x_k, b_k]$. If the two values happen to be equal, $f(x_k) = f(x_k')$, then the next interval can be either $[a_k, x_k]$ or $[x_k', b_k]$, since both intervals are of the same length $[b_k - x_k'] = [x_k - a_k]$. The last points of the process are given by

$$x_{N-1}' = (1/2+\varepsilon)(b_{N-1} - a_{N-1}) + a_{N-1}$$
$$x_{N-1} = 1/2\,(b_{N-1} - a_{N-1}) + a_{N-1} \qquad (9)$$

If we specify the maximum allowable error ε, then equations (9) yield unambiguously the minimum number of steps N, and vice versa, for a given number of experiments the limiting error in determining the point of extremum is

$$F_{n+1} \leq L/\xi \leq F_{n+2}$$

Here ξ is the error, L the interval length $F_{n+1} \leq \frac{1}{\xi} \leq F_{n+2}$ and F_{n+1}, F_{n+2} are Fibonacci numbers, and n the number of experiments. By way of example, here are twelve initial members of the Fibonacci series, 1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; 144. Hence, to attain a 5 per cent limit to the error (ξ =5 %), we would need 6 experimental steps. Indeed, $F_{6+1} < 20 < F_{6+2}$ or $13 < 20 < 21$. With 10 experiments, the error would drop below 0.69 %.

Multidimensional optimization of the fuse design can be done most efficiently with the aid of gradient methods. The great advantage of such techniques is that the derivatives of the optimality criterion need not be represented analytically. It is sufficient to calculate partial derivatives of the function at several individual points which can be done in terms of finite increments (both numerically and in an instrumental experiment). The coordinates of point $\bar{x}^{i+1}$ are calculated at each consequtive step of the procedure as

$$\bar{x}^{i+1} = \bar{x}^i + K(\bar{x}^i)\nabla f(\bar{x}^i), \quad i = 1, 2, \ldots, n \qquad (10)$$

where $\nabla f(\bar{x}^i) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is the gradient of $f(x_1, \ldots, x_n)$. It is a vector whose components are partial derivatives of $\bar{x}_i$, specifying the direction of motion from point $\bar{x}^i$ to $\bar{x}^{i+1}$. The proportionality matrix $K(\bar{x}^i)$ determines the iteration step length. A highly reliable method is Box-Wilson's steepest descent technique

within which the step length is chosen from the condition that the optimality criterion be maximum (minimum) along the gradient vector. In other words, the magnitude of K^i is found from

$$\max f[\bar{x}^i + K^i \nabla f(\bar{x}^i)] = f[\bar{x}^i + K^i \nabla f(\bar{x}^i)] = f(\bar{x}^{i+1}), \quad K^i > 0 \tag{11}$$

Gradient-based procedures of this kind employ linear approximations to $f(\bar{x})$, which allows evaluating first derivatives of the function, along with the function itself. This information is of great importance in the search of optimum. On the other hand, cycling of the process cannot be excluded. To improve the reliability, we can resort to Newton's method in which consequtive points are calculated from the recurrence relation

$$\bar{x}^{i+1} = \bar{x}^i - H^{-1}(\bar{x}^i) \nabla f(\bar{x}^i), \tag{12}$$

with $H(\bar{x}^i)$ denoting the second derivative matrix (i.e. the Hessian). The method proves particularly efficient near the optimum where the gradient is liable to changes in sign that result in sharp variations of the vector direction and deteriorate the procedure convergence.

Pontryagin's maximal principle. The maximal principle allows obtaining a solution even when other methods fail, e.g. in the presence of constraints and with implicitly specified function to be optimized. It suggests introduction of a set of variables $P_1(t), P_2(t), \ldots, P_n(t)$ for the dual problem, and a scalar Hamiltonian according to

$$H(x, y, P, t) = F(x, y, t) + \sum_{i=1}^{n} P_i(t) f_i(x, y, t) \tag{13}$$

Let the system under analysis be described by the equation

$$\frac{dx}{dt} = f[x(t), y(t), t] \tag{14}$$

with boundary conditions $x(t_0) = x_0$ and $x(T) = \mathcal{D}n$ and let it be necessary to find the control function $y(t)$ to minimize the functional $J = \int_{t_0}^{T} F(x, y, t)\, dt$. If the optimum control function $y(t)$ exists and belongs to the set $y(t) \in Y$, and $\dot{x}(t)$ is the optimum trajectory, then the dual state vector $\dot{P}(t)$ should and satisfy

$$\frac{\partial \dot{x}_i(t)}{\partial t} = \frac{\partial H}{\partial \dot{P}_i(t)} \quad \text{and} \quad \frac{\partial \dot{P}_i(t)}{\partial t} = -\frac{\partial H}{\partial \dot{x}_i(t)} \tag{15}$$

The maximal principle was used by the present writers to optimize the voltage waveform across the fuse arc.

The sensitivity analysis in fuse design suggests understanding the relations between design variables (such as structural parameters, external conditions and manufacture tolerances) and response of the structure, i.e. variations of its performance parameters. Suppose the equation to describe the structure behaviour is $K(b)z(b) = F(b)$, where $K(b)$ is the global rigidity matrix, $z(b)$ the solution, and $F(b)$ the load. In most cases the design problem is aimed at minimizing some cost function (or maximize the goal function) $\Psi[b, z(b)]$ in the presence of constraints on the design variables $b = [b_1, \ldots, b_k]$ and functional characteristics. Then the sensitivity analysis is oriented at finding

$$\frac{d\Psi}{db} = \frac{\partial \Psi}{\partial b} + \frac{\partial \Psi}{\partial z} K^{-1}(b) \frac{\partial}{\partial b}[F(b) - K(b)z] \tag{16}$$

This equation represents the direct differentiation method. A different approach is that of dual variables where the researcher seeks to evaluate

$$\frac{d\Psi}{db} = \frac{\partial \Psi}{\partial b} + \lambda^T \left[\frac{\partial F(b)}{\partial b} - \frac{\partial}{\partial b}(K(b)z)\right] \tag{17}$$

with $\lambda^T = K^{-1}(b) \frac{\partial \Psi^T}{\partial z}$ being the dual variable which is determined from the symmetry of K. Details of the calculations associated with the two approaches are described in [4]. The sensitivity analysis can be complementary to the identification and optimization procedures, either employing their results or accompanying them as a parallel process. Note that direct differentiation is efficient at the preliminary stage of analyzing model problems when the number of variables involved is low. In the majority of practical cases, the dual variable method is preferrable.

Gradient methods and the approaches used in sensitivity analysis are largely applicable to optimized parameter scaling for fuse structure modification. A complete recalculation of performance characteristics for new structure parameters can result, with the use of standard high-order models, in forbiddingly high costs and computer time expenditures. The volume of computations can be greatly reduced through the use of

results relating to a reference structure. E.g., let the basic physics in the reference structure be governed by the matrix equation $[K_o]\{\varphi_o\}=F_o$, where K_o is the conductivity matrix (alternatively, the rigidity matrix) specified by the structure parameters, and φ_o the potential (temperature, or mechanical stress) matrix representing response of the structure to the applied load F_o (electrical, thermal or mechanical). By modifying the structure, the designer changes the matrix K_1, which leads to the necessity of finding the new response φ_1 through the solution of the equation $[K_1]\{\varphi_1\}=F_1$. It is possible, however, to make use of the knowledge obtained from the reference equation, and scale the response by generating the series of vectors $\tilde{\varphi}^{(0)}, \tilde{\varphi}^{(1)}, \tilde{\varphi}^{(2)}$, etc. as

$$\tilde{\varphi}^{(i+1)} = \tilde{\varphi}_o - [K_o]^{-1}\Delta K\,\tilde{\varphi}^{(i)} \qquad (18)$$

with $\Delta K = K_1 - K_o$ and $\tilde{\varphi}^{(0)} = \varphi_o$

With moderate (up to 10%) changes in the structure parameters, the sequence $\tilde{\varphi}^{(0)}$, $\tilde{\varphi}^{(1)}, \tilde{\varphi}^{(2)}$, etc. converges to the solution φ_1. Since the best convergence is provided by rational approximating functions, these can be used to obtain a solution with as high parameter variations as 35 to 47 %. The required response is sought in the form of the rational function

$$\varphi(x) = [K_o(x) + x_o K(x)]\,F.$$

Conclusion. The approaches discussed allow the problem of fuse design and analysis
i) to be formulated in rigorous mathematical terms for specific practical cases;
ii) to be analyzed on a systematic basis, as a consistent set of logically interdependent algorithms;
iii) to be treated by efficient and appropriate methods.

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