MATHEMATICAL ANALYSIS OF BREAKING PERFORMANCE OF CURRENT- LIMITING FUSES

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ONE-HALF-CYCLE FUSING CURRENT Pre-arcing Joule-integral of a fuse-link for sufficiently large over-currents can be treated as a constant specific to that fuse-link, irrespective whether it is of current-limiting or of expulsion type. Many of the calculations hitherto made on pre-arcing phenomena have been based on this empirical law.

In the following analyses, this constant value of the pre-arcing Joule-integral shall be represented by "one-half-cycle fusing current" $I_{T/2}$ [A] of the fuse-link, as introduced by S.B. Toniolo and G. Cantarella. It is defined by

(1) $$I_{T/2}^2 \cdot T/2 = K$$

where T denotes one period of the power frequency in seconds and K means the constant Joule-integral of the fuse-link in $A^2 \cdot s$. Re-writing (1), we get

(2) $$\begin{aligned} I_{T/2} &= 10\sqrt{K} \quad \text{for 50 Hz, and} \\ &= 11\sqrt{K} \quad \text{for 60 Hz.} \end{aligned}$$

In most cases, these values are nearly equal to one-half-cycle fusing current on the time-current characteristic; however, for semi-conductor fuses and for micro-fuses, these values are often considerably smaller than the current given by the time-current characteristic.

Also, in the followings, all currents in a.c. and d.c. circuits shall be expressed in multiples of $I_{T/2}$ of the fuse-link to be put in the circuit. This manupulation makes the results of mathematical analyses applicable to wide varieties of circumstances: a single formula or a single diagram will now be able to cover wide range of prospective current combined with varieties of ratings of fuse-links.

MAKING ANGLE DIAGRAM When the source voltage $e=\sqrt{2}E \sin(\theta+\psi)$, where $\theta= 2\pi t/T$, is applied to a fused circuit of power factor angle ϕ at the instant t=0 or θ=0, the following current will flow in the circuit:

(3) $$\bar{i} = \sqrt{2}\bar{I}[\sin(\theta+\psi-\phi) - \sin(\psi-\phi)\varepsilon^{-\theta/\tan\phi}]$$

where both the instantaneous value of current and the prospective current are expressed in terms of $I_{T/2}$ of the fuse-link in the circuit and the short lines over the symbols indicate that these quantities have been made

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dimension-less by dividing by $I_{T/2}$. The first term on the right side of (3) gives a.c. current which will remain after the second term has subsided, lagging angle ϕ behind the voltage wave; the second term, canceling the first term at $\theta=0$, attenuates with time constant $L/R = T\cdot\tan\phi/2\pi$.

When the prospective current of the circuit is large enough, the fuse will operate giving the constant value of the pre-arcing Joule-integral, i.e.

$$\int_0^{\theta_0-\psi} \bar{i}^2 d\theta = \pi \tag{4}$$

where θ_0 is the arc-initiation angle of the fuse-link on the voltage wave. Putting (3) into (4), we get[Appendix A]

$$\begin{aligned}\bar{I}^2 = \pi[&\theta_0-\psi-\tfrac{1}{2}\sin 2(\theta_0-\phi)+\sin(\psi-\phi)\cos(\psi+2\phi)/\cos\phi \\ &+4\sin\theta_0\sin\phi\sin(\psi-\phi)\varepsilon^{-(\theta_0-\psi)/\tan\phi} \\ &-\sin^2(\psi-\phi)\tan\phi\varepsilon^{-2(\theta_0-\psi)/\tan\phi}]^{-1}\end{aligned} \tag{5}$$

Eq.(5) is rather complicated, but it can be re-written in the simplified form of

$$\bar{I} = f(\theta_0,\psi,\phi) \tag{6}$$

which indicates that there exists a definite relationship among the prospective current in terms of $I_{T/2}$, arc-initiation angle Θ_0, making angle ψ on the voltage wave, and the power factor angle ϕ. Thus, if two parameters are specified, the relationship between the other two factors can be represented in a single diagram.

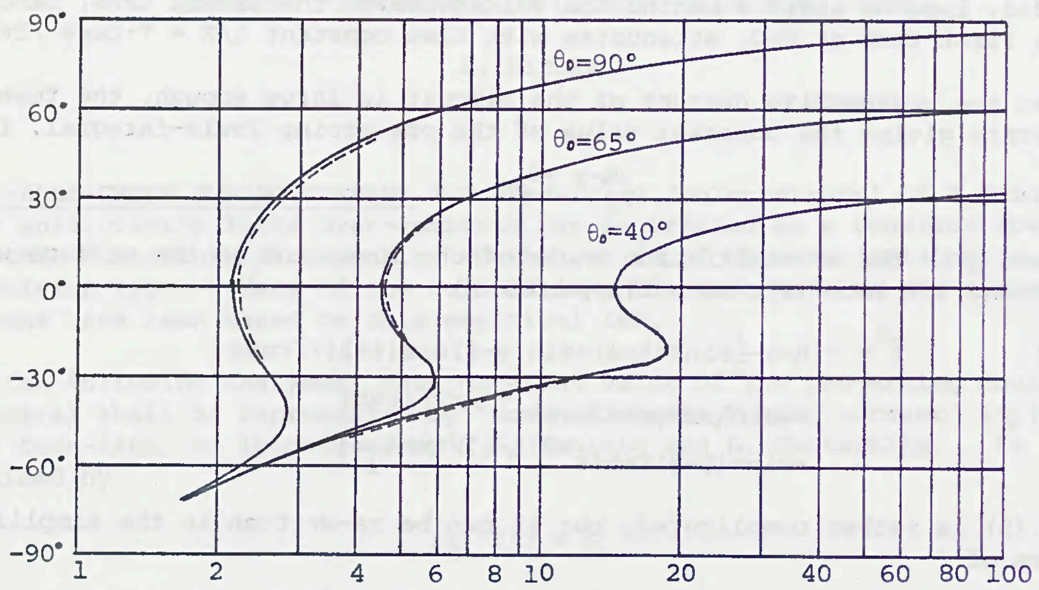
IEC Publications 269-1 and 282-1 prescribe regarding the breaking capacity tests that the arc-initiation angle θ_0 shall be from 40° to 65° for one test and from 65° to 90° for two tests. In answer to this requirement, Diagram 1 on top of the following page has been prepared, which represents the computed relationship between the making angle ψ and the breaking current in terms of $I_{T/2}$, taking the arc-initiation angle θ_0 as parameter and assuming the power factor of the test circuit as 0.05. The dotted lines are for power factor 0.15, which indicates the effect of the power factor is so small.

This Diagram indicates that for breaking capacity tests for $\bar{I} = 50$, the making angle of the test circuit should be chosen at about 45° for one test and at about 70° for two tests. This is applicable also for the case where the test frequency is 60 Hz, provided $I_{T/2}$, or the divisor of $\bar{I}$, is correctly calculated from (2).

It is noted that the IEC specification for the arc-initiation between 40° and 65° is impossible to be met for the test currents less than 4.5 in $\bar{I}$. Furthermore, if the current were less than 2.2, even the confinement of the arc-initiation between 65° and 90° becomes practically impossible.

<u>GENERALIZED CUTOFF CHARACTERISTIC FOR A.C.</u> Cutoff current of a current-limiting fuse-link having wire-elements, usually, coincides with the instantaneous current in the circuit at the instant of the arc-initiation. Thus, the latter should be examined in place of the cutoff current.

Diagram 1



So-called "cutoff characteristic" is composed of two parts: the lower inclined part where the cutoff may take place, and the higher inclined part where the cutoff never occurs. In any case, the cutoff current for a given prospective current is the maximum let-through current that could flow in a circuit for the most unfavourable making angle.

First the lower inclined portion should be studied. As stated before, the cutoff current of wire-element fuse-links can be deemed as equal to the instantaneous current $\bar{i}_o$ at the arc-initiation, which is given by substituting $\theta+\psi=\theta_o$ into (3). However, it should be simplified in the form,

$$\bar{i}_o = f_1(\bar{I},\psi,\theta_o,\phi).$$

This means that the instantaneous current $\bar{i}_o$ at the arc-initiation is to be determined by making angle ψ and the arc-initiation angle θ_o for given values of $\bar{I}$ and ϕ. On the other hand, ψ and θ_o are related with each other just for the same values of $\bar{I}$ and ϕ by means of (5) or Diagram 1. Thus, it is possible to get the relationship,

(7) $$\bar{i}_o = f_2(\bar{I},\psi,\phi).$$

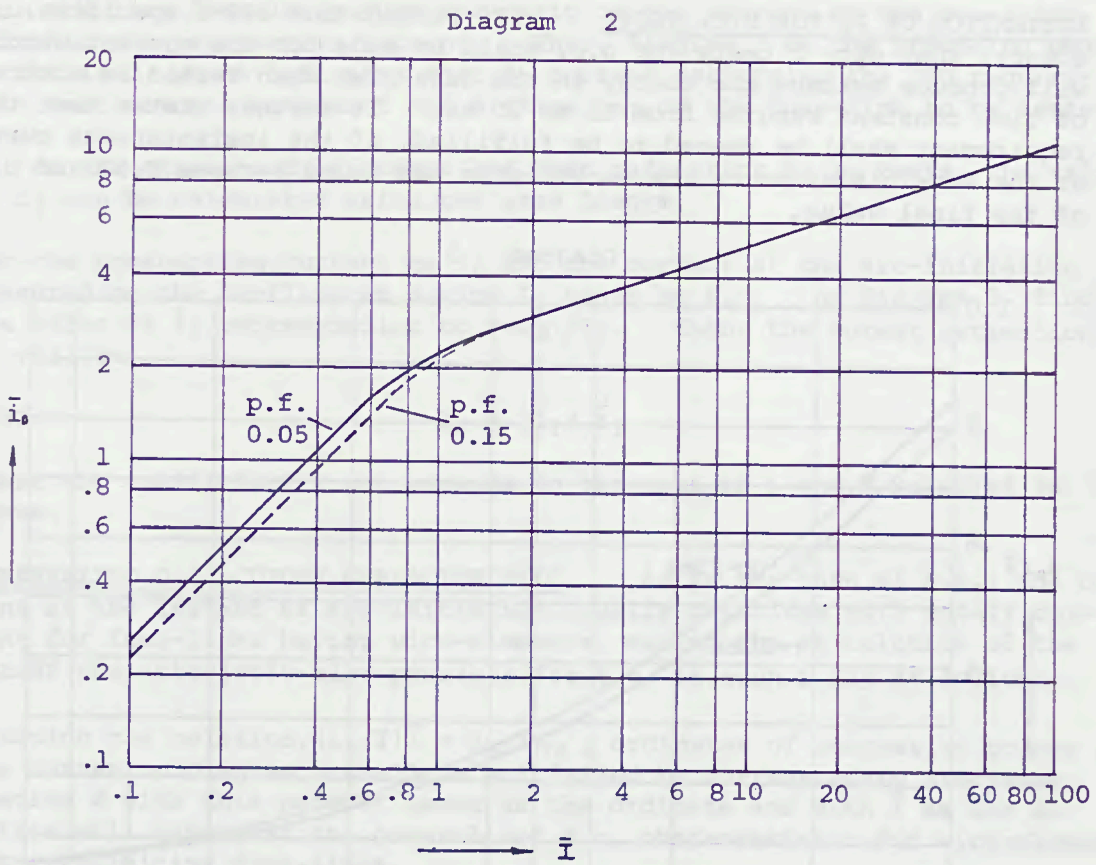
Since the cutoff current indicated on the cutoff characteristic is the maximum value, which corresponds to the most unfavourable making angle,

(8) $$\partial\bar{i}_o/\partial\psi = 0.$$

Combining (7) and (8) and eliminating ψ, we obtain ultimately

(9) $$\bar{i}_o = F(\bar{I},\phi)$$

which just indicates the lower inclined portion of the a.c. cutoff characteristic of wire-element fuse-links. Diagram 2 on the next page shows the result by a digital computer. [Appendix B]



For the highly inclined portion, where cutoff never takes place, the computation should start directly from (3) or from its simplified form

$$\bar{i} = g(\bar{I},\psi,\theta,\phi).$$

Now, for given $\bar{I}$ and ϕ, the instantaneous value of current is a function of ψ and θ. Thus, the maximum let-through current must satisfy the following two conditions:

$$\partial\bar{i}/\partial\psi = 0$$

$$\partial\bar{i}/\partial\theta = 0$$

Combining all three equations, we get

(10) $$\bar{i}_{max} = G(\bar{I},\phi)$$

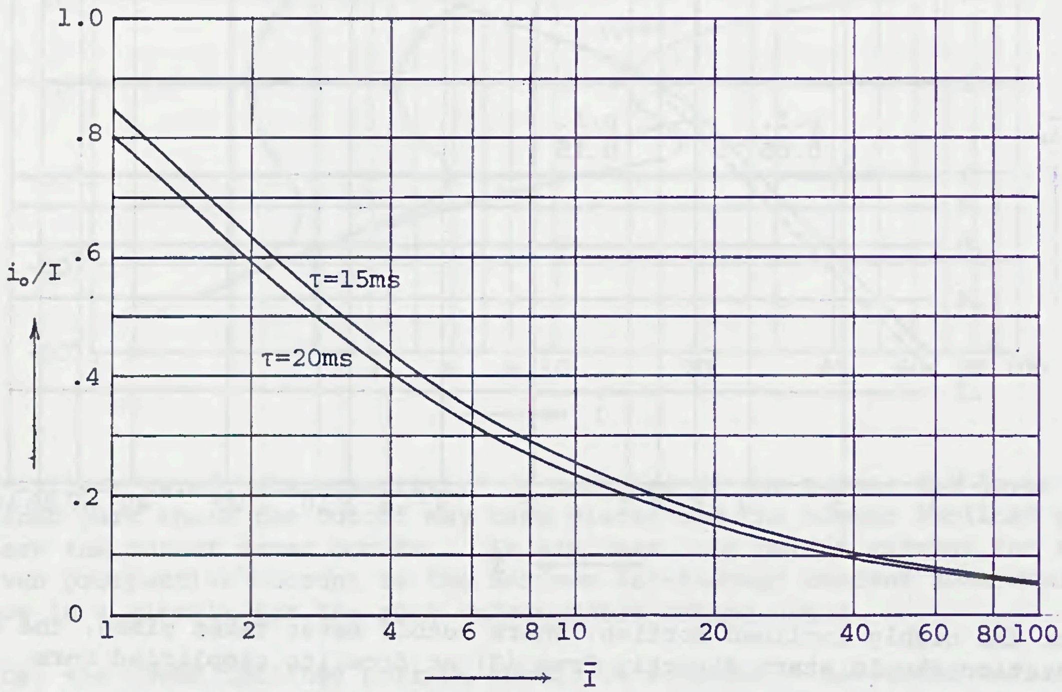
which represents the highly inclined portion of the Diagram.

It would be note-worthy that this portion is essentially independent of fuse-links, because it indicates the current in the circuit before the arc initiates in the fuse-link.

Though the direct application of this Diagram is limited to the wire-element fuse-links, it will also serve as a guidance in preparing cutoff characteristics of strip-element fuse-links.

ESTIMATION OF I_2 FOR D.C. TESTS   IEC-Publication 269-1 specifies, in effect, that duty 2 tests for d.c. shall be made for the current which will produce maximum arc energy in the fuse-link when tested in a circuit of time constant ranging from 15 to 20 ms. It further states that this requirement shall be deemed to be fulfilled, if the instantaneous current at the arc-initiation on the oscillogram was found between 0.50 and 0.80 of the final value.

Diagram 3



However, some estimation of the value of I_2 would be welcome to engineers in the test laboratories. This can be made as follows:

Let the rising d.c. current be represented by

(11) $$i = I(1 - \varepsilon^{-t/\tau})$$

where I is the d.c. prospective current and τ is the time constant of the test circuit.

The arc will start at the instant when the pre-arcing Joule-integral of the fuse-link reached its specific constant value. Thus, the following equation applies:

$$I^2 \int_o^{t_o} (1 - \varepsilon^{-t/\tau})^2 \, dt = I^2_{1/2} \cdot T/2$$

where t_o and τ denote the pre-arcing time and the time constant, respectively.

Making the integration and re-arranging the result, we obtain [Appendix C]

(12) $$\bar{I} = \left(\frac{- T/\tau}{2 \ln(1 - \gamma) + 2\gamma + \gamma^2}\right)^{0.5}$$

where $\bar{I} = I/I_{1/2}$, $\gamma = i_o/I$

γ in the last formula is just the ratio of the current at the arc-initiation to the prospective current. Thus, Diagram 3 on the preceding page is obtained, which indicates that I_2 current satisfying the IEC requirement must situate between 1 and 3 times $I_{\tau/2}$ of the fuse-link to be tested.

If, on the other hand, I_1 tests had been made prior to I_2 tests, the value of I_2 can be calculated using the same Diagrm.

Let the prospective current be I_1 and the current at the arc-initiation measured on the oscillogram during I_1 tests be i_{o1}. On Diagram 3, find the value of $\bar{I}_1$ corresponding to $\gamma = i_{o1}/I_1$. Then, the surest estimation of I_2 will be

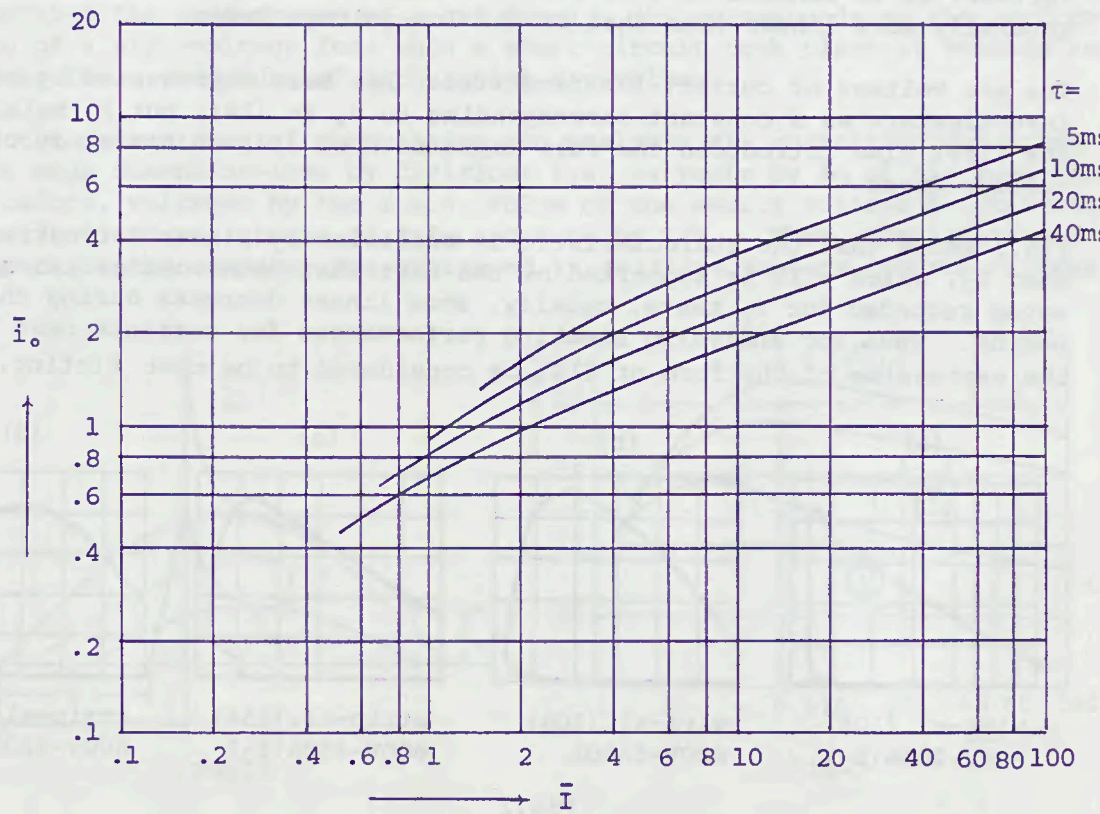
(13) $$I_2 = 2I_1/\bar{I}_1$$

where the coefficient 2 corresponds to the mean of 1 and 3 referred to above.

<u>GENERALIZED D.C. CUTOFF CHARACTERISTIC</u> As in the case of a.c., the current at the instant of arc-initiation usually coincides with cutoff current for fuse-links having wire-elements, making the calculation of the cutoff characteristic also possible for d.c. in such a way as follows:

Noticing the relation, $(i_o/I)\bar{I} = i_o/I_{\tau/2}$, ordinates of successive points on the curves on Diagram 3 shall be multiplied by corresponding abscissae. Diagram 4 with this product taken as the ordinate and with $\bar{I}$ as the abscissa will give just the generalized d.c. characteristic for wire-element current-limiting fuse-links.

Diagram 4



It should be noted that compared to a.c. cutoff characteristic where power factor of the circuit played a minor role, d.c. cutoff current is appreciably affected by the time constant of the circuit.

<u>REPRESENTATION OF V-I CHARACTERISTIC OF ARCS IN C.L. FUSES</u> The arc confined in sand has a peculiar V-I characteristic compared to the arcs in open air or in gases. The small area of the tunnel left behind the vapourization of fuse-element confines the arc so narrowly as to give the arc very high current-density. At the same time, sand granules on the inner wall of the tunnel cools the arc so intensely by fusion and vapourization that the arc voltage is appreciably higher than that of the free arc.

Another distinguished feature of the arc in sand is its positive resistance characteristic as indicated in the oscillograms in Fig. 1, where voltage across the fuse-terminals was taken as ordinate and current as abscissa. Referring to the oscillogram (a), current starts at the bottom left and goes right horizontally to the bottom right, where the fusion occurs and the voltage jumps up to the top right. This voltage spike limits the current, and both current and voltage decrease along the nearly straight line down to left, where some residual voltage is recognized.

These oscillograms suggest us that the V-I characteristic of c.l. fuses might be represented by

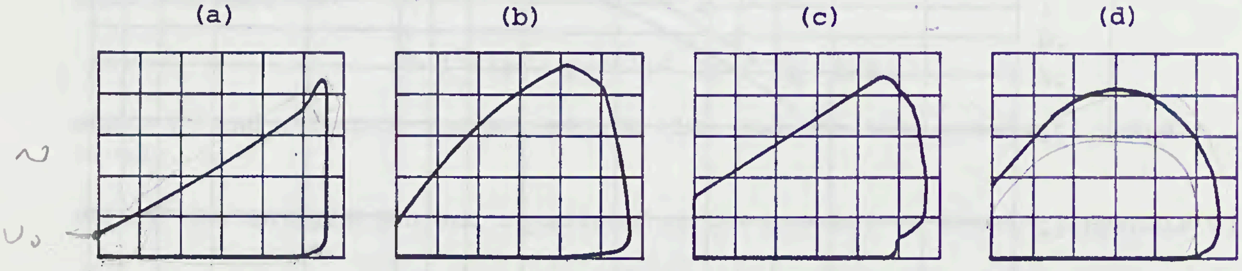
(14) $$v_a = v_o + ri$$

for fairly wide range of breaking currents.

In our experience the V-I characteristic of low-voltage wire-element fuses are more linear than that of fuses having strip-elements with many holes. Further, it is considered that the characteristic of high-voltage fuses are generally more linear than that of the low-voltage fuses.

The arc voltage of current-limiting fuses has been represented by many investigaters as a constant corresponding to v_o in (14), but F. Meier for the first time introduced the full expression of (14) in his study of low-voltage fuses.

Fig.1 shows that the characteristic is sufficiently linear for currents near I_2, which will be supported by the fact that both voltage and current waves recorded for I_2 tests, usually, show linear decrease during the arc period. Thus, for analysing breaking performances for currents near I_2, the expression of the form of (14) is considered to be most fitting.



(a) wire-el. (10A) 600V-200A (I_2)

(b) wire-el. (10A) 600V-5600A

(c) strip-el. (15A) 600V-480A (I_2)

(d) strip-el. (15A) 600V-5600A

i

Fig.1

SIMULATION OF THE BREAKING PERFORMANCE BY ANALOGUE COMPUTER The breaking performance of a current-limiting fuse, when the arc voltage v_a is expressed by (14), can be reduced simply to the problem of the circuit theory. Furthermore, it can even be simulated by an analogue computer which operates in principle, as follows:

After closing the switch S_1 in Fig.2 at a pre-set making angle ψ, the 2nd power of the current through the circuit is integrated by the computer till it reaches the pre-arcing Joule-integral of the fuse-link to be tested. The computer stops here and the switch S_2 is opened to introduce the arc voltage into the circuit. The computer re-starts and begins to integrate the product of v_a and i, until the current reaches zero. The voltage and the current waves throughout the computation are recorded by the pen-oscillograph.

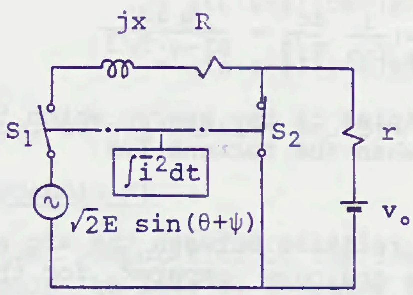


Fig.2

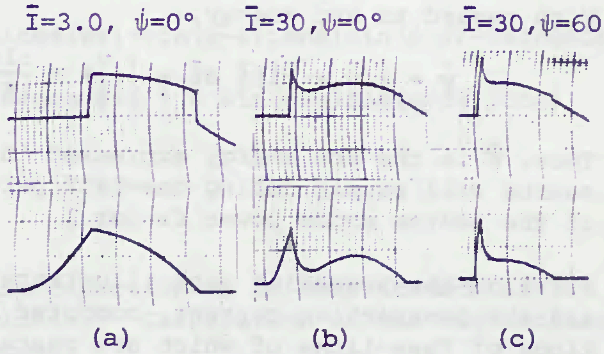


Fig.3

Some of the wave forms furnished by the computer are shown in Fig.3. (a), corresponding to the I_2 test, shows the upward swell of the current wave which indicates that the given arc voltage was too small. The computer simulated the performance of a bad fuse! (b) corresponds to the performance of a high-voltage fuse when a short-circuit took place at voltage zero. (c) is for a good fuse of rather high arc voltage.

For the convenience of computation and analysis all quantities concerned were made dimension-less by division: i.e. currents by $I_{T/2}$ of the fuse-link as before, voltages by the r.m.s. value of the source voltage E, resistance by a fictious resistance of $E/I_{T/2}$ and time by T/2. Thus, the inputs and outputs of the computer are expressed in relative numbers, source voltage

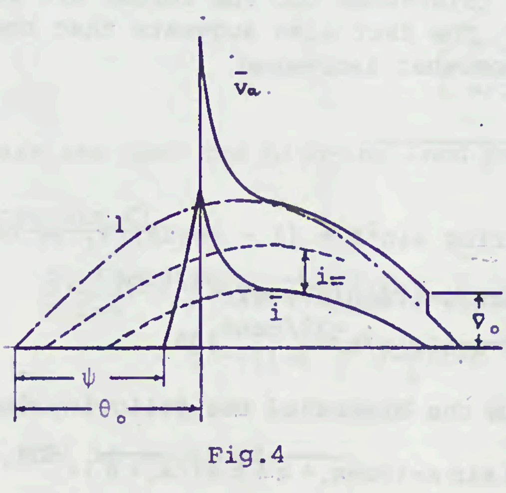


Fig.4

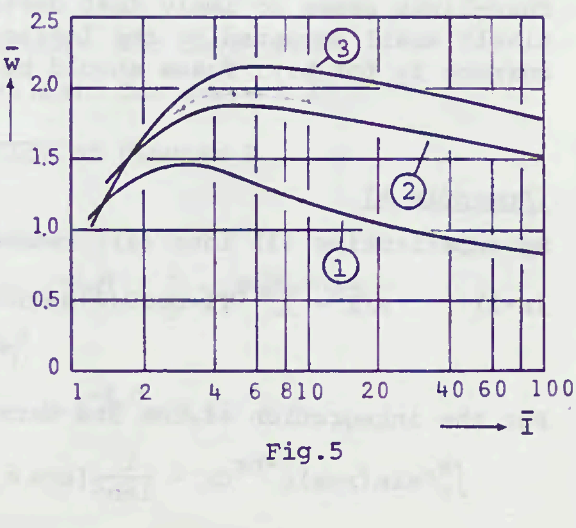


Fig.5

expressed by 1. In the followings all the quantities made dimension-less are indicated by a short line over corresponding symbols.

Thus, e.g. arc voltage is expressed by

$$\frac{v_a}{E} = \frac{v_o}{E} + \frac{rI_{T/2}}{E}\cdot\frac{i}{I_{T/2}}$$

or by

$$\bar{v}_a = \bar{v}_o + \bar{r}\cdot\bar{i}$$

where $\bar{v}_a$ is the ratio of the arc voltage to the r.m.s. value of the source voltage E and $\bar{r}$ corresponds to the voltage drop in terms of the source voltage E, in a resistor of r[Ω] carrying current $I_{T/2}$.

With regard to arc energy,

$$\bar{w} = \int(\bar{v}_o + \bar{r}\bar{i})\bar{i}\ d\bar{t} = \int\left(\frac{v_o}{E} + \frac{rI_{T/2}}{E}\cdot\frac{i}{I_{T/2}}\right)\frac{i}{I_{T/2}}\frac{dt}{T/2} = \frac{\int v_a\,idt}{EI_{T/2}\cdot T/2}$$

Thus, $\bar{W}$ is the arc energy expressed in multiples of the energy which the source will supply during one-half period, when the current $I_{T/2}$ of the source under power factor 1.

Fig.5 on the preceding page illustrates the relation between the arc energy and the prospective current, computed by the analogue computer, for three kinds of fuse-links of which arc characteristics are shown in the Table.

fuse-links	#1	#2	#3
value of $\bar{v}_o$	0.60	0.35	0.35
value of $\bar{r}$	0.70	0.50	0.40

The values of $\bar{v}_o$ and $\bar{r}$ for #1 corresponds to the V-I characteristic of a l.v. fuse-link, while the other two are for h.v. fuses.

The computation indicates that the arc energy increases with the decrease in the values of $\bar{v}_o$ and $\bar{r}$ and that the current corresponding to the maximum arc energy moves rightwards. It also indicates that at the same time the top of the hill becomes flat. This is partly supported by the fact that the arc energy of a l.v. fuse increases when tested under increased voltages, which signifies values of $\bar{v}_o$ and $\bar{r}$ decrease in inverse proportion to E or the test voltage.

The fact that h.v. fuse-links have, in general, lower $\bar{v}_o$ and $\bar{r}$ than l.v. fuse-links seems to imply that design tolerances for the former are relatively small compared to the latter. The fact also suggests that the test current I_2 for h.v. fuses should be somewhat increased.

[Appendix A]

By substituting (3) into (4), remembering $\sin^2 A = (1 - \cos 2A)/2$, we have

(A-1) $$\pi/\bar{I}^2 = \int_0^{\theta_o-\psi}[1-\cos 2(\theta+\psi-\phi) - 4\sin(\psi-\phi)\sin(\theta+\psi-\phi)\varepsilon^{-\theta/\tan\phi} + (1-\cos 2\psi-\phi)\varepsilon^{-2\theta/\tan\phi}]d\theta$$

For the integration of the 3rd term in the brackets, the following formula

$$\int_0^{x_o}\sin(x+a)\varepsilon^{-bx}dx = \frac{1}{1+b^2}[\cos a + b\sin a - (\cos\overline{x_o+a} + b\sin\overline{x_o+a})\varepsilon^{-bx_o}]$$

should be consulted. Substituting $x=\theta$, $x_o=\theta_o-\psi$, $a=\psi-\phi$ and $b=1/\tan\phi$ in the formula and remembering $\sin(A+B)=\sin A\cos B+\sin B\cos A$, we have

$$\int_o^{\theta_o-\psi}\sin(\theta+\psi-\phi)\,\varepsilon^{-\theta/\tan\phi}d\theta = \sin\phi[\sin\psi-\sin\theta_o\,\varepsilon^{-(\theta_o-\psi)/\tan\phi}]$$

Integrating also the other terms in brackets in (A-1), we get

$$\text{(A-2)}\quad \pi/\bar{I}^2=\theta_o-\psi-\tfrac{1}{2}\sin2(\theta_o-\phi)+[\tfrac{1}{2}\sin2(\psi-\phi)-4\sin\psi\sin\phi\sin(\psi-\phi)+\tan\phi\sin^2(\psi-\phi)]$$
$$+4\sin\theta_o\sin\phi\sin(\psi-\phi)\,\varepsilon^{-(\theta_o-\psi)/\tan\phi}-\tan\phi\sin^2(\psi-\phi)\,\varepsilon^{-2(\theta_o-\psi)/\tan\phi}$$

Terms between the brackets in (A-2) can be re-arranged as follows:

$$\tfrac{1}{2}\sin2(\psi-\phi)-4\sin\psi\sin\phi\sin(\psi-\phi)+\tan\phi\sin^2(\psi-\phi)$$
$$=\sin(\psi-\phi)[\cos(\psi-\phi)-2\sin\psi\sin\phi]+\sin(\psi-\phi)\tan\phi[\sin(\psi-\phi)-2\sin\psi\cos\phi]$$
$$=\sin(\psi-\phi)[\cos(\psi+\phi)-\tan\phi\sin(\psi+\phi)] = \sin(\psi-\phi)\cos(\psi+2\phi)/\cos\phi$$

Thus, we get finally Eq.(5).

[Appendix B]

Actual computation of the low-inclined portion is so complicated that it should be left to computer specialists. Calculation of the high-inclined portion, however, is relatively simple.

$$\text{Since,}\quad \partial\bar{i}/\partial\theta = \sqrt{2}\,\bar{I}[\cos(\theta+\psi-\phi)+\sin(\psi-\phi)\,\varepsilon^{-\theta/\tan\phi}/\tan\phi] = 0$$

$$\text{and}\quad \partial\bar{i}/\partial\psi = \sqrt{2}\,\bar{I}[\cos(\theta+\psi-\phi)-\cos(\psi-\phi)\,\varepsilon^{-\theta/\tan\phi}] = 0$$

Subtracting the 2nd from the 1st equation, we get

$$\text{(B-1)}\quad \sqrt{2}\bar{I}[\sin(\psi-\phi)/\tan\phi+\cos(\psi-\phi)]\,\varepsilon^{-\theta/\tan\phi}=0$$

or $\quad\sin\psi = 0$, or $\quad\psi = 0$

Putting this into (B-1), $\cos(\theta-\phi) = \cos\phi\,\varepsilon^{-\theta/\tan\phi}$ is obtained.

Solving this equation with respect to ψ for given values of ϕ, and substituting these values of ψ and ϕ into Eq.(3), we get

$$\bar{i}=\sqrt{2}\bar{I}(1.86)\quad\text{for } \cos\phi= 0,05$$
$$\bar{i}=\sqrt{2}\bar{I}(1.63)\quad\text{for } \cos\phi= 0.15$$

These are just the high-inclined portion of Diagram 2.

[Appendix C]

$$\frac{2}{T}\left(\frac{I}{I_{T/2}}\right)^2=\int_o^{t_o}(1-\varepsilon^{-t/\tau})^2\,dt = t_o-2\tau(1-\varepsilon^{-t_o/\tau})+\frac{\tau}{2}(1-\varepsilon^{-2t_o/\tau})$$
$$=t_o-\tau(1-\varepsilon^{-t_o/\tau})-\frac{\tau}{2}(1-\varepsilon^{-t_o/\tau})^2$$

Since $\quad -t_o/\tau = \ln(1-i_o/I)$, and $\quad 1-\varepsilon^{-t_o/\tau}= i_o/I = \gamma$

Eq.(12) is obtained.